

On the generalized sum of squared logarithms inequality

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Abstract

Assume $n \geq 2$. Consider the elementary symmetric polynomials $e_k(y_1, y_2, \dots, y_n)$ and denote by E_0, E_1, \dots, E_{n-1} the elementary symmetric polynomials in reverse order

$$E_k(y_1, y_2, \dots, y_n) := e_{n-k}(y_1, y_2, \dots, y_n) = \sum_{i_1 < \dots < i_{n-k}} y_{i_1} y_{i_2} \dots y_{i_{n-k}}, \quad k \in \{0, 1, \dots, n-1\}.$$

Let moreover S be a nonempty subset of $\{0, 1, \dots, n-1\}$. We investigate necessary and sufficient conditions on the function $f: I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, such that the inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq f(b_1) + f(b_2) + \dots + f(b_n) \quad (*)$$

holds for all $a = (a_1, a_2, \dots, a_n) \in I^n$ and $b = (b_1, b_2, \dots, b_n) \in I^n$ satisfying

$$E_k(a) < E_k(b) \text{ for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \dots, n-1\} \setminus S.$$

As a corollary, we obtain (*) if $2 \leq n \leq 4$, $f(x) = \log^2 x$ and $S = \{1, \dots, n-1\}$, which is the sum of squared logarithms inequality previously known for $2 \leq n \leq 3$.

Key words: elementary symmetric polynomials, logarithm, matrix logarithm, inequality, characteristic polynomial, invariants, positive definite matrices, inequalities

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1 Introduction - the sum of squared logarithms inequality

In a previous contribution [1] the sum of squared logarithms inequality has been introduced and proved for the particular cases $n = 2, 3$. For $n = 3$ it reads: let $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ be given positive numbers such that

$$\begin{aligned} a_1 + a_2 + a_3 &\leq b_1 + b_2 + b_3, \\ a_1 a_2 + a_1 a_3 + a_2 a_3 &\leq b_1 b_2 + b_1 b_3 + b_2 b_3, \\ a_1 a_2 a_3 &= b_1 b_2 b_3. \end{aligned}$$

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Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 \leq \log^2 b_1 + \log^2 b_2 + \log^2 b_3.$$

The general form of this inequality can be conjectured as follows.

Definition 1.1

The standard elementary symmetric polynomials e_1, \dots, e_{n-1}, e_n are

$$e_k(y_1, \dots, y_n) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} y_{j_1} \cdot y_{j_2} \cdot \dots \cdot y_{j_k}, \quad k \in \{1, 2, \dots, n\}; \quad (1.1)$$

note that $e_n = y_1 \cdot y_2 \cdot \dots \cdot y_n$.

Conjecture 1.2 (Sum of squared logarithms inequality)

Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be given positive numbers. Then the condition

$$e_k(a_1, \dots, a_n) \leq e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n-1\}, \quad e_n(a_1, \dots, a_n) = e_n(b_1, \dots, b_n)$$

implies that

$$\sum_{i=1}^n \log^2 a_i \leq \sum_{i=1}^n \log^2 b_i.$$

Remark 1.3

Note that the conclusions of Conjecture 1.2 are trivial provided we have equality everywhere, i.e.

$$e_k(a_1, \dots, a_n) = e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n\}. \quad (1.2)$$

In this case, the coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ are equal up to permutations, which can be seen by looking at the characteristic polynomials of two matrices with eigenvalues a_1, \dots, a_n and b_1, \dots, b_n . From this perspective, having equality just in the last product e_n and strict inequality else seems to be the most difficult case.

Based on extensive random sampling on \mathbb{R}_+^n for small numbers n it has been conjectured that Conjecture 1.2 might be true for arbitrary $n \in \mathbb{N}$. The sum of squared logarithms inequality has immediate important applications in matrix analysis ([8], see also [3]) as well as in nonlinear elasticity theory [5, 6, 7, 4]. In matrix analysis it implies that the global minimizer over all rotations to

$$\inf_{Q \in \text{SO}(n)} \|\text{sym}_* \text{Log } Q^T F\|^2 = \|\sqrt{F^T F}\|^2 \quad (1.3)$$

at given $F \in \text{GL}^+(n)$ is realized by the orthogonal factor $R = \text{polar}(F)$ (such that $R^T F = \sqrt{F^T F}$). Here, $\|X\|^2 := \sum_{i,j=1}^n X_{ij}^2$ denotes the Frobenius matrix norm and $\text{Log} : \text{GL}(n) \rightarrow \mathfrak{gl}(n) = \mathbb{R}^{n \times n}$ is the multivalued matrix-logarithm, i.e. any solution $Z = \text{Log } X \in \mathbb{C}^{n \times n}$ of $\exp(Z) = X$ and $\text{sym}_*(Z) = \frac{1}{2}(Z^* + Z)$.

Recently, the case $n = 2$ was used to verify the polyconvexity condition in nonlinear elasticity [6, 5] for a certain class of isotropic energy functions. For more background information on the sum of squared logarithms inequality we refer the reader to [1].

In this paper we extend the investigation as to the validity of Conjecture 1.2 by considering arbitrary functions f instead of $f(x) = \log^2 x$. We formulate this more general problem and we are able to extend Conjecture 1.2 to the case $n = 4$. The same methods should also be useful for proving the statement for $n = 5, 6$. However, the necessary technicalities prevent us from discussing these cases in this paper.

In addition, we present ideas which might be helpful in attacking the fully general case, namely arbitrary f and arbitrary n .

2 The generalized inequality

In order to generalize Conjecture 1.2 in the directions hinted at in the introduction, we consider from now on a non-standard definition of the elementary symmetric polynomials. In fact, for $n \geq 2$ it will be more convenient for us to reverse their numbering and define E_0, E_1, \dots, E_{n-1} by

$$E_k(y_1, \dots, y_n) := e_{n-k}(y_1, \dots, y_n) = \sum_{i_1 < \dots < i_{n-k}} y_{i_1} \cdot y_{i_2} \cdot \dots \cdot y_{i_{n-k}}, \quad k \in \{0, 1, \dots, n-1\}. \quad (2.1)$$

In particular, now

$$\begin{aligned} E_0(y_1, \dots, y_n) &:= e_n(y_1, \dots, y_n) = y_1 \cdot y_2 \cdot \dots \cdot y_n, \\ E_{n-1}(y_1, \dots, y_n) &:= e_1(y_1, \dots, y_n) = y_1 + y_2 + \dots + y_n. \end{aligned} \quad (2.2)$$

Let $I \subset \mathbb{R}$ be an open interval and let

$$\Delta_n := \{y = (y_1, y_2, \dots, y_n) \in I^n \mid y_1 \leq y_2 \leq \dots \leq y_n\}. \quad (2.3)$$

Let S be a nonempty subset of $\{0, 1, \dots, n-1\}$ and assume that $a, b \in \Delta_n$ are such that

$$E_k(a) < E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \dots, n-1\} \setminus S. \quad (2.4)$$

In this section we investigate necessary and sufficient conditions for a (smooth) function $f: I \rightarrow \mathbb{R}$, such that the inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq f(b_1) + f(b_2) + \dots + f(b_n)$$

holds for all $a, b \in \Delta_n$ satisfying assumption (2.4).

Remark 2.1

The formulation of the above problem has a certain monotonicity structure: we assume that “ $E(a) < E(b)$ ” and want to prove that “ $F(a) < F(b)$ ”. Therefore our idea is to consider a curve y connecting the points a and b , such that $E(y(t))$ “increases”. Then the function $g(t) = F(y(t))$ should also increase and therefore $g'(t) > 0$ must hold. From this we are able to derive necessary and sufficient conditions on the function f .

This approach motivates the following definition.

Definition 2.2 (b dominates a , $a \preceq b$)

Let $a, b \in \Delta_n$. We will say that b dominates a , and denote $a \preceq b$, if there exists a piecewise differentiable mapping $y: [0, 1] \rightarrow \Delta_n$ (i.e. y is continuous on $[0, 1]$ and differentiable in all but at most countably many points) such that $y(0) = a$, $y(1) = b$, $y_i(t) \neq y_j(t)$ for $i \neq j$ and all but at most countably many $t \in [0, 1]$ and the functions

$$A_k(t) := E_k(y(t)), \quad k \in \{0, 1, \dots, n-1\}$$

are non-decreasing on the interval $[0, 1]$.

If $a \preceq b$, then $E_k(a) = A_k(0) \leq A_k(1) = E_k(b)$, so it follows from Definition 2.2 that a, b satisfy assumption (2.4) with S being the set of all k for which $A_k(t)$ is not a constant function on $[0, 1]$.

We are ready to formulate the main results of this section.

Theorem 2.3

Assume that $a, b \in \Delta_n$ and let $a \preceq b$. Let $S \subseteq \{0, 1, \dots, n-1\}$ denote the set of all integers k with $E_k(a) < E_k(b)$. Moreover, assume that $f \in C^n(I)$ be such that

$$(-1)^{n+k} (x^k f'(x))^{(n-1)} \leq 0 \quad \text{for all } x \in I \text{ and all } k \in S. \quad (2.5)$$

Then the following inequality holds:

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq f(b_1) + f(b_2) + \dots + f(b_n). \quad (2.6)$$

A partially reverse statement is also true.

Theorem 2.4

Let $f \in C^n(I)$ be such that the inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq f(b_1) + f(b_2) + \dots + f(b_n) \quad (2.7)$$

holds for all $a, b \in \Delta_n$ satisfying

$$E_k(a) \leq E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \dots, n-1\} \setminus S \quad (2.8)$$

for some subset $S \subseteq \{0, 1, \dots, n-1\}$. Then f satisfies property (2.5), i.e.

$$(-1)^{n+k} (x^k f'(x))^{(n-1)} \leq 0 \quad \text{for all } x \in I \text{ and all } k \in S. \quad (2.9)$$

In this respect, we can formulate another conjecture:

Conjecture 2.5

Let S be a nonempty subset of $\{0, 1, \dots, n-1\}$ and assume that $a, b \in \Delta_n$ are such that (2.4) is satisfied, i.e.

$$E_k(a) < E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \dots, n-1\} \setminus S.$$

Then there exists a curve y satisfying the conditions from Definition 2.2 and thus $a \preceq b$.

Remark 2.6

In concrete applications of Theorem 2.3 and Theorem 2.4 one would like to know whether condition (2.4) already implies $a \preceq b$. This is Conjecture 2.5. Unfortunately, we are able to prove Conjecture 2.5 only for $2 \leq n \leq 4$, $I = (0, \infty)$ and $S \subseteq \{1, 2, \dots, n-1\}$ (see the next section).

Example 2.7

It is easy to see that if $I = (0, \infty)$ then the function $f(x) = \log^2 x$ satisfies property (2.5) for $S = \{1, 2, \dots, n-1\}$. Indeed, we proceed by induction on n . For $n = 2$ and $k = 1$ the property is immediate. Moreover, for $k \geq 2$ and $n \geq 3$ we get

$$\begin{aligned} (-1)^{n+k} (x^k f'(x))^{(n-1)} &= 2(-1)^{n+k} (x^{k-1} \log x)^{(n-1)} \\ &= 2(-1)^{n+k} ((k-1)x^{k-2} \log x)^{(n-2)} + 2(-1)^{n+k} (x^{k-2})^{(n-2)} \leq 0 \end{aligned} \quad (2.10)$$

by the induction hypothesis, since the second summand vanishes. It remains to check property (2.5) for $k = 1$, which is also immediate.

Note also that property (2.5) is not true for $k = 0$. Therefore Theorem 2.3 and Theorem 2.4 for $f(x) = \log^2 x$ attain the following formulation:

Corollary 2.8

Assume that $a, b \in \mathbb{R}_+^n$ be such that $a \preceq b$ and $a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$. Then

$$\log^2(a_1) + \log^2(a_2) + \dots + \log^2(a_n) \leq \log^2(b_1) + \log^2(b_2) + \dots + \log^2(b_n)$$

and this inequality fails, if the constraint $a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$ is replaced by the weaker one $a_1 a_2 \dots a_n \leq b_1 b_2 \dots b_n$.

In order to see that the weaker condition is not sufficient for the inequality to hold, consider the case

$$a = \left(\frac{1}{n}, \dots, \frac{1}{n}\right), \quad b = (1, \dots, 1).$$

Then $a \preceq b$ and $a_1 a_2 \dots a_n \leq b_1 b_2 \dots b_n$, but

$$\log^2(a_1) + \log^2(a_2) + \dots + \log^2(a_n) = n \log^2(n) > 0 = \log^2(b_1) + \log^2(b_2) + \dots + \log^2(b_n).$$

Remark 2.9

Corollary 2.8 is a weaker statement than Conjecture 1.2 since we assume that $a \preceq b$. If Conjecture 2.5 is true, then Conjecture 1.2 follows.

Example 2.10

The function $f(x) = x^p$ ($x > 0$) with $p \in (0, 1)$ satisfies property (2.5) for the set $S = \{0, 1, \dots, n-1\}$. Indeed, for each $n \geq 2$ and $0 \leq k \leq n-1$, we have

$$(-1)^{n+k}(x^k f'(x))^{(n-1)} = (-1)^{n+k} p(k+p-1)(k+p-2) \dots (k+p-(n-1)) x^{k+p-n}.$$

The above product is not greater than 0, because among the factors $k+p-1, k+p-2, \dots, k+p-(n-1)$ there are exactly $n-1-k$ negative ones.

Similarly, the function $f(x) = x^p$ for $p \in (-1, 0)$ satisfies property (2.5) for the set $S = \{1, 2, \dots, n-1\}$, because $p < 0$ and among the factors $k+p-1, k+p-2, \dots, k+p-(n-1)$ there are exactly $n-k$ negative ones. On the other hand, property (2.5) is not true for $k=0$.

Thus, similarly like above, we have

Corollary 2.11

Assume that $a, b \in (0, \infty)^n$ be such that $a \preceq b$ and $a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$. If $p \in (-1, 1)$, then

$$a_1^p + a_2^p + \dots + a_n^p \leq b_1^p + b_2^p + \dots + b_n^p.$$

This inequality fails for $-1 < p < 0$ (but remains true for $0 < p < 1$) if the constraint $a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$ is replaced by the weaker one $a_1 a_2 \dots a_n \leq b_1 b_2 \dots b_n$.

Proof of Theorem 2.3 If S is empty, then $E_k(a) = E_k(b)$ for all $k \in \{0, 1, \dots, n-1\}$ and hence $a = b$, which immediately implies the inequality. We therefore assume that S is nonempty.

Let $y: [0, 1] \rightarrow \Delta_n$ be the curve connecting points a and b as in Definition 2.2. Consider the function

$$\begin{aligned} p(t, x) &= (x + y_1(t))(x + y_2(t)) \dots (x + y_n(t)) = \sum_{k=0}^{n-1} x^k E_k(y(t)) + x^n \\ &= (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k \in S} x^k A_k(t), \end{aligned} \quad (2.11)$$

where $A_k(t) = E_k(y(t)) - E_k(a)$ is a non-decreasing mapping. Our goal is to show that the function

$$\eta(t) = \sum_{i=1}^n f(y_i(t)) \quad (2.12)$$

is non-decreasing on $[0, 1]$, i.e. we show that $\eta'(t) \geq 0$ a.e. on $(0, 1)$.

To this end, fix $i \in \{1, 2, \dots, n\}$. Since $p(t, -y_i(t)) = 0$ for all $t \in (0, 1)$, we obtain

$$\partial_1 p(t, -y_i(t)) + \partial_2 p(t, -y_i(t)) \cdot (-y_i'(t)) = 0$$

for all $t \in (0, 1)$ and therefore

$$\sum_{k \in S} (-y_i(t))^k A_k'(t) + \prod_{j \neq i} (y_j(t) - y_i(t)) \cdot (-y_i'(t)) = 0, \quad (2.13)$$

which gives

$$y_i'(t) = \sum_{k \in S} (-y_i(t))^k A_k'(t) \left(\prod_{j \neq i} (y_j(t) - y_i(t)) \right)^{-1}.$$

This equality holds, if $y_i(t) \neq y_j(t)$ for $i \neq j$, which is true for all but countably many values of $t \in (0, 1)$. For those values of t we get

$$\begin{aligned} \eta'(t) &= \sum_{i=1}^n f'(y_i(t)) \cdot y_i'(t) \\ &= \sum_{i=1}^n f'(y_i(t)) \cdot \sum_{k \in S} (-y_i(t))^k A_k'(t) \left(\prod_{j \neq i} (y_j(t) - y_i(t)) \right)^{-1} \\ &= \sum_{k \in S} A_k'(t) \sum_{i=1}^n f'(y_i(t)) \cdot (-y_i(t))^k \left(\prod_{j \neq i} (y_j(t) - y_i(t)) \right)^{-1}. \end{aligned} \quad (2.14)$$

Fix $t \in (0, 1)$ such that $y_i(t) \neq y_j(t)$ for $i \neq j$ and write $y_i = y_i(t)$ for simplicity. Since $A'_k(t) \geq 0$, we will be done, if we show that

$$\widehat{D} := \sum_{i=1}^n f'(y_i) \cdot (-y_i)^k \left(\prod_{j \neq i} (y_j - y_i) \right)^{-1} \geq 0 \quad \text{for all } k \in S.$$

To this end, consider the polynomial

$$g(x) = \sum_{i=1}^n f'(y_i) \cdot (-y_i)^k \left(\prod_{j \neq i} (y_j - y_i) \right)^{-1} \cdot \prod_{j \neq i} (x - y_j).$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to \widehat{D} . Moreover,

$$g(y_i) = f'(y_i) \cdot (-y_i)^k \cdot (-1)^{n-1} \quad (i = 1, 2, \dots, n).$$

Therefore the function $h(x) = g(x) + (-1)^{n+k} x^k f'(x)$ has n different roots y_1, y_2, \dots, y_n in the interval I . It follows that the function

$$h^{(n-1)}(x) = (n-1)! \widehat{D} + (-1)^{n+k} (x^k f'(x))^{(n-1)} \quad (2.15)$$

has a root in the interval I , and since $(-1)^{n+k} (x^k f'(x))^{(n-1)} \leq 0$ for all $x \in I$, it follows that $\widehat{D} \geq 0$, which completes the proof of Theorem 2.3. \blacksquare

Proof of Theorem 2.4 Suppose, to the contrary, that $(-1)^{k+n} (x^k f'(x))^{(n-1)} > 0$ for some $x \in I$ and some $k \in S$. Then $(-1)^{k+n} (x^k f'(x))^{(n-1)} > 0$ holds for all x belonging to some interval J contained in I . Choose the numbers $a_1 < a_2 < \dots < a_n$ from J and consider

$$p(t, x) = (x + a_1) \cdot (x + a_2) \cdot \dots \cdot (x + a_n) + t x^k.$$

Then for all sufficiently small t ($0 < t < \varepsilon$), there exist different numbers $y_i(t)$ belonging to J , such that

$$p(t, x) = (x + y_1(t))(x + y_2(t)) \dots (x + y_n(t)).$$

Then

$$x^n + \sum_{i=0}^{n-1} E_i(a) \cdot x^i + t x^k = p(t, x) = x^n + \sum_{i=0}^{n-1} E_i(y(t)) \cdot x^i,$$

and since $t > 0$, we see that a and $b = y(t)$ satisfy (2.8). We will be done if we show that

$$f(a_1) + f(a_2) + \dots + f(a_n) > f(y_1(t)) + f(y_2(t)) + \dots + f(y_n(t)).$$

We proceed in the same way as in the proof of Theorem 2.3. We define

$$\eta(t) = \sum_{i=1}^n f(y_i(t)) \quad \text{for } 0 < t < \varepsilon$$

and this time we want to show that $\eta'(t) < 0$ for $0 < t < \varepsilon$.

By the Inverse Mapping Theorem (see proof of Proposition 3.4 below for a more detailed explanation), $y \in C^1(0, \varepsilon)$ and therefore

$$\eta'(t) = \sum_{i=1}^n f'(y_i(t)) \cdot y'_i(t) = \sum_{i=1}^n f'(y_i(t)) \cdot (-y_i(t))^k \left(\prod_{j \neq i} (y_j(t) - y_i(t)) \right)^{-1}. \quad (2.16)$$

Now, like previously, write $y_i = y_i(t)$ for simplicity. Our goal is therefore to prove that

$$\widehat{D} := \sum_{i=1}^n f'(y_i) \cdot (-y_i)^k \left(\prod_{j \neq i} (y_j - y_i) \right)^{-1} < 0.$$

Consider the polynomial

$$g(x) = \sum_{i=1}^n f'(y_i) \cdot (-y_i)^k \left(\prod_{j \neq i} (y_j - y_i) \right)^{-1} \cdot \prod_{j \neq i} (x - y_j).$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to \widehat{D} . Moreover, the function $h(x) = g(x) + (-1)^{n+k}x^k f'(x)$ has n different roots y_1, y_2, \dots, y_n in the interval J . It follows that the function

$$h^{(n-1)}(x) = (n-1)! \widehat{D} + (-1)^{n+k}(x^k f'(x))^{(n-1)}$$

has a root in the interval J . And since $(-1)^{n+k}(x^k f'(x))^{(n-1)} > 0$ for all $x \in J$, it follows that $\widehat{D} < 0$, which completes the proof of Theorem 2.4. ■

3 Construction of the connecting curve

In this section we prove that condition (2.4) implies $a \preceq b$, if $2 \leq n \leq 4$, $I = (0, \infty)$ and $S \subseteq \{1, 2, \dots, n-1\}$. However, we start with a construction of the desired curve for a general interval I , integer $n \geq 2$ and set $S \subseteq \{0, 1, \dots, n-1\}$.

For $a, b \in \Delta_n$, we say that $a < b$, if $a \neq b$ and $E_k(a) \leq E_k(b)$ for all $k = 0, 1, \dots, n-1$. We say that $a \leq b$, if $a < b$ or $a = b$.

Definition 3.1

For $a < b$ denote by $\mathcal{C}(a, b)$ the set of all piecewise differentiable (i.e. continuous and differentiable in all but at most countably many points) curves y in Δ_n satisfying:

- (a) the curve $y(t)$ starts at a (i.e. $y(0) = a$, if the curve $y(t)$ is parametrized by the interval $[0, \varepsilon]$);
- (b) $y(t) \in \text{int}(\Delta_n)$ for all but at most countable many values t ;
- (c) the mappings $E_k(y(t))$ are non-decreasing in t and $E_k(y(t)) \leq E_k(b)$ for all t and each $k = 0, 1, \dots, n-1$.

Note that a curve in $\mathcal{C}(a, b)$ does not necessarily end at the point b .

Proposition 3.2

Let $n \geq 2$ be a positive integer and let S be a nonempty subset of $\{0, 1, \dots, n-1\}$. Let moreover $a, b \in \Delta_n$ be such that (2.4) holds. Furthermore, suppose that for all $c \in \Delta_n$ with $a \leq c < b$ the set $\mathcal{C}(c, b)$ is nonempty. Then $a \preceq b$.

Proof. Each element (curve) of $\mathcal{C}(a, b)$ is a (closed) subset of Δ_n . We equip the set $\mathcal{C}(a, b)$ with the inclusion relation \subseteq , obtaining a nonempty partially ordered set $(\mathcal{C}(a, b), \subseteq)$. We are going to show that each chain $\{y_i\}_{i \in \mathcal{I}}$ has an upper bound in $\mathcal{C}(a, b)$.

To achieve this, consider the curve

$$y_0 = \overline{\bigcup_{i \in \mathcal{I}} y_i},$$

i.e. the concatenation of the curves y_i . Then obviously y_0 satisfies conditions (a) and (c) of Definition 3.1. To prove (b) assume that y_0 is parametrized on $[0, 1]$. Then for each positive integer k the curve y_k , defined as the restriction of y_0 to the interval $[0, 1 - \frac{1}{k}]$, is contained in some curve $y_i \in \mathcal{C}(a, b)$ of the given chain $\{y_i\}$. Therefore $y_k(t)$ is piecewise differentiable and satisfies condition (b) for each positive integer k . Moreover,

$$y_0 = \overline{\bigcup_{k=1}^{\infty} y_k}.$$

Hence y_0 is piecewise differentiable and satisfies (b) as well.

Now, by the Kuratowski-Zorn lemma, there exists a maximal element y in $(\mathcal{C}(a, b), \subseteq)$. We show that y is a desired curve connecting the points a and b , which will imply that $a \preceq b$.

To this end, it is enough to show that, if the curve y is parametrized on $[0, 1]$, then $y(1) = b$. Suppose, to the contrary, that $y(1) = c \neq b$. Then $a \leq c < b$, and hence the set $\mathcal{C}(c, b)$ is nonempty. Thus the curve y can be extended beyond the point c , which contradicts the fact that y is a maximal element in $\mathcal{C}(a, b)$. This completes the proof of Proposition 3.2. ■

From now on assume that $I = (0, \infty)$ and S is a nonempty subset of $\{1, 2, \dots, n-1\}$.

In order to prove that (2.4) implies $a \preceq b$, it suffices to show that the sets $\mathcal{C}(a, b)$ for $a, b \in \Delta_n$ with $a < b$ are nonempty. This is implied by the following conjecture, which we will prove later for $n \leq 4$.

Conjecture 3.3

Let $n \geq 2$ be an integer and $a \in \Delta_n$. Let S be a nonempty subset of $\{1, 2, \dots, n-1\}$ with the property that there exist $A_k > 0$ for $k \in S$ such that all the roots of the polynomial

$$q(x) = (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k \in S} A_k x^k$$

are real (and hence negative). Then there exist continuous on $[0, \varepsilon]$, differentiable on $(0, \varepsilon)$ and nondecreasing mappings $B_k : [0, \varepsilon] \rightarrow \mathbb{R}$ ($k \in S$) with $B_k(0) = 0$ such that $\sum_{k \in S} B_k(t)$ is increasing on $[0, \varepsilon]$ and for all sufficiently small values of $t > 0$ the polynomial

$$(x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k \in S} B_k(t)x^k$$

has n distinct real (and hence negative) roots.

Now we show how Conjecture 3.3 implies that the sets $\mathcal{C}(a, b)$ are nonempty.

Proposition 3.4

Let n and S be such that the conjecture holds. Let moreover $a, b \in \Delta_n$ be such that (2.4) holds. Then the set $\mathcal{C}(a, b)$ is nonempty.

Proof. Consider the polynomials

$$p(x) = (x + a_1)(x + a_2) \dots (x + a_n) \quad \text{and} \quad q(x) = (x + b_1)(x + b_2) \dots (x + b_n).$$

Then

$$q(x) - p(x) = \sum_{k=0}^{n-1} (E_k(b) - E_k(a))x^k = \sum_{k \in S} A_k x^k,$$

where $A_k > 0$ for all $k \in S$. According to the conjecture, there exist continuous on $[0, \varepsilon]$ and differentiable on $(0, \varepsilon)$ nondecreasing mappings $B_k : [0, \varepsilon] \rightarrow \mathbb{R}$, with $B_k(0) = 0$ such that $\sum_{k \in S} B_k(t)$ is increasing on $[0, \varepsilon]$ and for all $t \in (0, \varepsilon)$ the polynomial

$$p(x) + \sum_{k \in S} B_k(t)x^k$$

has n distinct real (and hence negative) roots $-y_n(t) < -y_{n-1}(t) < \dots < -y_1(t) < 0$. We show that $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ defines a differentiable curve (parametrized on $[0, \varepsilon]$) that belongs to $\mathcal{C}(a, b)$, provided ε is chosen in such a way that $B_k(\varepsilon) \leq A_k$ for $k \in S$.

Consider the mapping $\Psi : \overline{\Delta_n} \rightarrow \Psi(\overline{\Delta_n})$ given by

$$\Psi(y) = (E_{n-1}(y), E_{n-2}(y), \dots, E_0(y)).$$

Then it follows from Remark 1.3 that the mapping Ψ is injective, hence Ψ is a continuous bijection defined on a closed subset of \mathbb{R}^n . Therefore the restriction $\Psi|_U$ of Ψ to a neighbourhood U of a is continuously invertible and thus

$$y(t) = \Psi^{-1}(\Psi(a) + (B_0(t), B_1(t), \dots, B_{n-1}(t))) \quad (t \in [0, \varepsilon])$$

(here we put $B_k(t) = 0$ for $k \notin S$) is a curve starting at a ; note that $\Psi(a) + (B_0(t), B_1(t), \dots, B_{n-1}(t))$ is contained in $\Psi(U)$ for sufficiently small ε . Moreover $y(t) \in \Delta_n$. Hence condition (a) is satisfied. Since $y(t) \in \text{int}(\Delta_n)$ for all $t \in (0, \varepsilon)$, condition (b) holds. It is also clear that (c) is satisfied, since $E_k(y(t)) = E_k(a) + B_k(t) \leq E_k(a) + A_k = E_k(b)$ for all $k \in \{0, 1, \dots, n-1\}$.

It remains to prove that $y(t)$ is differentiable on $(0, \varepsilon)$. This however is a consequence of the Inverse Mapping Theorem, if we show that

$$\det[D\Psi(y)] \neq 0 \quad \text{for all } y \in \text{int}(\Delta_n).$$

To this end, let $V(y)$ be the $n \times n$ Vandermonde-type matrix given by $V_{ij}(y) = (-y_i)^{n-j}$ ($1 \leq i, j \leq n$). This matrix is obtained from the standard Vandermonde matrix

$$W(-y_1, -y_2, \dots, -y_n) = \begin{pmatrix} 1 & -y_1 & (-y_1)^2 & \dots & (-y_1)^{n-1} \\ 1 & -y_2 & (-y_2)^2 & \dots & (-y_2)^{n-1} \\ 1 & -y_3 & (-y_3)^2 & \dots & (-y_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -y_n & (-y_n)^2 & \dots & (-y_n)^{n-1} \end{pmatrix} \quad (3.1)$$

by reversing the order of columns of W .

Since [2]

$$(D\Psi(y))_{jk} = \frac{\partial}{\partial y_k} E_{n-j}(y) = \begin{cases} 1 & : j = 1, \\ E_{n-j}(y^{(k)}) & : j > 1, \end{cases}$$

where $y^{(k)} = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$ is y with its k -th component removed, it follows from the general formula

$$t^{n-1} + \sum_{j=0}^{n-2} t^j E_j(z_1, z_2, \dots, z_{n-1}) = (t + z_1)(t + z_2) \dots (t + z_{n-1}) \quad (3.2)$$

that

$$\begin{aligned} (V(y) \cdot D\Psi(y))_{ik} &= \sum_{j=1}^n (V(y))_{ij} \cdot (D\Psi(y))_{jk} \\ &= (-y_i)^{n-1} + \sum_{j=2}^n (-y_i)^{n-j} \cdot E_{n-j}(y^{(k)}) \\ &= (-y_i)^{n-1} + \sum_{j=0}^{n-2} (-y_i)^j \cdot E_j(y^{(k)}) = \prod_{j \neq k} (y_j - y_i) \end{aligned}$$

and thus

$$V(y) \cdot D\Psi(y) = \text{diag} \left(\prod_{j \neq 1} (y_j - y_1), \prod_{j \neq 2} (y_j - y_2), \dots, \prod_{j \neq n} (y_j - y_n) \right). \quad (3.3)$$

It is well-known that

$$\det[V(y)] = \prod_{i < j} (y_j - y_i) \neq 0 \quad (y \in \text{int } \Delta_n).$$

Therefore we obtain

$$\det[D\Psi(y)] = \prod_{i < j} (y_i - y_j) \neq 0 \quad (y \in \text{int } \Delta_n),$$

which completes the proof of Proposition 3.4. ■

Lemma 3.5

Assume that $n \geq 3$ is odd and let $0 < a_1 \leq a_2 \leq \dots \leq a_n$. Let moreover $A_k \geq 0$ for $k = 1, 2, \dots, (n-1)/2$ with at least one A_k not equal to 0. Consider the polynomials

$$\begin{aligned} P(x) &= (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k-1}, \\ Q(x) &= (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k}. \end{aligned} \quad (3.4)$$

Then the polynomial P has exactly one root in the interval $(-a_1, 0)$ and at most two roots in the interval $(-a_n, -a_{n-1})$. Moreover, the polynomial Q has exactly one root in the interval $(-\infty, -a_n)$ and at most two roots in the interval $(-a_2, -a_1)$.

Proof. That P has exactly one root in $(-a_1, 0)$ follows immediately from the observation that $P(-a_1) < 0$, $P(0) > 0$ and $P'(x) > 0$ on $(-a_1, 0)$.

Now we show that Q has exactly one root in $(-\infty, -a_n)$.

Dividing the equation $Q(x) = 0$ by $x^n a_1 a_2 \dots a_n$ and substituting $z = 1/x$ and $b_i = 1/a_i$, yields the equation $P_0(z) = 0$, where

$$P_0(z) = (z + b_1)(z + b_2) \dots (z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k-1}$$

for some nonnegative numbers B_k , not all equal to 0. We already know that P_0 has exactly one root in the interval $(-b_n, 0)$, so it follows that Q has exactly one root in the interval $(-\infty, -a_n)$.

Now we prove that Q has at most two roots in the interval $(-a_2, -a_1)$. To the contrary, suppose that Q has at least 3 roots in $(-a_2, -a_1)$. Since $Q(-a_2) > 0$ and $Q(-a_1) > 0$, it follows that Q has an even number, and hence at least four, roots in the interval $(-a_2, -a_1)$.

Let $0 > -c_1 \geq -c_2 \geq \dots \geq -c_{n-1}$ be the roots of $p'(x) = 0$, where

$$p(x) = (x + a_1)(x + a_2) \dots (x + a_n). \quad (3.5)$$

Then $a_1 < c_1 < a_2$. The polynomial $Q(x)$ is decreasing on the interval $[-a_2, -c_1]$, so it has at most one root in this interval. Therefore the polynomial Q has at least three roots in the interval $(-c_1, -a_1)$, and consequently the equation $Q''(x) = 0$ has a root in $(-c_1, -a_1)$. But $Q''(x) > 0$ for all $x > -c_1$, a contradiction. Hence Q must have at most two roots in $(-a_2, -a_1)$.

Finally, to prove that P has at most two roots in the interval $(-a_n, -a_{n-1})$, divide the equation $P(x) = 0$ by $x^n a_1 a_2 \dots a_n$ and substitute $z = 1/x$ and $b_i = 1/a_i$. This reduces to the equation $Q_0(z) = 0$, where

$$Q_0(z) = (z + b_1)(z + b_2) \dots (z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k}$$

for some nonnegative numbers B_k , not all equal to 0. We already know that Q_0 has at most two roots in the interval $(-b_{n-1}, -b_n)$, so it follows that P has at most two roots in the interval $(-a_n, -a_{n-1})$. This completes the proof of Lemma 3.5. \blacksquare

The same proof yields an analogous result for even values of n .

Lemma 3.6

Assume that $n \geq 2$ is even and let $0 < a_1 \leq a_2 \leq \dots \leq a_n$. Let moreover $A_k \geq 0$ for $k = 1, 2, \dots, n/2$ and not all of the A_k 's are equal to 0. Consider the polynomials

$$\begin{aligned} P(x) &= (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k=1}^{n/2} A_k x^{2k-1}, \\ Q(x) &= (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k=1}^{n/2-1} A_k x^{2k}. \end{aligned} \quad (3.6)$$

Then the polynomial P has exactly one root in each of the intervals $(-\infty, -a_n)$ and $(-a_1, 0)$ and Q has at most two roots in each of the intervals $(-a_n, -a_{n-1})$ and $(-a_2, -a_1)$.

Proof. The same proof as that for Lemma 3.5 can be used. \blacksquare

Now we turn to the proof of Conjecture 3.3 for $2 \leq n \leq 4$ and an arbitrary nonempty set $S \subseteq \{1, 2, \dots, n-1\}$.

We first make some useful general remarks.

Let $I(a) = \{i \in \{1, 2, \dots, n-1\} : a_i = a_{i+1}\}$. If $I(a)$ is empty, then the conjecture holds. Indeed, if $k \in S$, then all the roots of the polynomial

$$(x + a_1)(x + a_2) \dots (x + a_k) + t x^k$$

are, for all sufficiently small $t > 0$, real and distinct.

On the other hand, if $I(a) = \{1, 2, \dots, n-1\}$, then only the set $S = \{1, 2, \dots, n-1\}$ possibly satisfies the assumptions of the conjecture. Indeed, suppose that $l \notin S$ and let $-b_1 \geq -b_2 \geq \dots \geq -b_n$ be the roots of

$$q(x) = (x + a_1)^n + \sum_{k \in S} A_k x^k.$$

Then by the inequality of arithmetic and geometric means, we obtain

$$\frac{E_l(a)}{\binom{n}{l}} = \frac{E_l(b)}{\binom{n}{l}} \geq (E_0(b))^{(n-l)/n} = (E_0(a))^{(n-l)/n} = \frac{E_l(a)}{\binom{n}{l}}, \quad (3.7)$$

and hence $b_1 = b_2 = \dots = b_n$. Since $E_0(a) = E_0(b)$, it follows that $a = b$, i.e. $A_k = 0$ for all $k \in S$. A contradiction.

Let I be a non-empty subset of $\{1, 2, \dots, n-1\}$. We observe that the conjecture is true for a set S and all $a \in \Delta_n$ with $I(a) = I$, if it is true for a set $T = \{n-k : k \in S\}$ and all $b \in \Delta_n$ with $I(b) = \{n-i : i \in I\}$. Indeed: if all the roots of the polynomial

$$q(x) = (x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k \in S} A_k x^k$$

are real, then substituting $x = 1/z$ and $a_i = 1/b_i$, we infer that all the roots of the polynomial

$$r(z) = (z + b_1)(z + b_2) \dots (z + b_n) + \sum_{l \in T} B_l z^l$$

are real. Hence there exist continuous on $[0, \varepsilon]$, differentiable on $(0, \varepsilon)$ and nondecreasing mappings $C_l(t)$ with $C_l(0) = 0$ such that the polynomial

$$(z + b_1)(z + b_2) \dots (z + b_n) + \sum_{l \in T} C_l(t) z^l$$

has n distinct real roots. Substituting $z = 1/x$ and $b_i = 1/a_i$, we infer that the polynomial

$$(x + a_1)(x + a_2) \dots (x + a_n) + \sum_{k \in S} C_{n-k}(t) x^k$$

has n distinct real roots.

For $n = 2$ the only possibility for the set S is $\{1\}$ and it is enough to notice that the polynomial $(x + a_1)(x + a_2) + tx$ has two distinct real roots for any $t > 0$.

Assume now $n = 3$. Then, in view of the above remarks, we have to consider two cases: 1) $a_1 < a_2 = a_3$; 2) $a_1 = a_2 = a_3$.

1) If $2 \notin S$, then the condition of Conjecture 3.3 can not be satisfied since for $A_1 > 0$, according Lemma 3.5, the polynomial

$$P(x) = (x + a_1)(x + a_2)^2 + A_1 x$$

has only one real root in the interval $(-a_1, 0)$ and obviously no roots on $\mathbb{R} \setminus (-a_1, 0)$. Thus P has only one real root for all $A_1 > 0$. We can therefore assume $2 \in S$, and for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2 + tx^2$$

has three distinct real roots.

2) According to the above remarks, $S = \{1, 2\}$. Then the polynomial $(x + a_1)^3 + ta_1x + tx^2$ has 3 distinct real roots for all sufficiently small $t > 0$.

Assume $n = 4$. In this case we have 5 possibilities: 1) $a_1 = a_2 < a_3 < a_4$; 2) $a_1 < a_2 = a_3 < a_4$; 3) $a_1 < a_2 = a_3 = a_4$; 4) $a_1 = a_2 < a_3 = a_4$; 5) $a_1 = a_2 = a_3 = a_4$.

1) We note that $S \neq \{2\}$, since, by Lemma 3.6, the polynomial

$$Q(x) = (x + a_1)^2(x + a_3)(x + a_4) + A_2 x^2 \quad \text{for } A_2 > 0$$

has at most two real roots in the interval $(-a_4, -a_3)$ and obviously no roots on $\mathbb{R} \setminus (-a_4, -a_3)$. Thus Q has at most two real roots. Therefore S contains an odd integer k . Then for all sufficiently small $t > 0$, the polynomial $(x + a_1)^2(x + a_3)(x + a_4) + tx^k$ has four distinct real roots.

2) Note that $2 \in S$, since by Lemma 3.6, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + A_1 x + A_3 x^3 \quad \text{for } A_1, A_3 > 0$$

has at most two real roots. Then for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + tx^2$$

has four distinct real roots.

3) We observe that $\{1, 2\} \subset S$ or $\{2, 3\} \subset S$, since by Lemma 3.6, each of the polynomials

$$(x + a_1)(x + a_2)^3 + A_1x + A_3x^3 \quad \text{and} \quad (x + a_1)(x + a_2)^3 + A_2x^2 \quad \text{for } A_1, A_2, A_3 > 0$$

as well as

$$(x + a_1)(x + a_2)^3 + A_1x \quad \text{and} \quad (x + a_1)(x + a_2)^3 + A_3x^3 \quad \text{for } A_1, A_3 > 0$$

has at most two real roots. Moreover, we prove that $S \neq \{1, 2\}$.

Suppose that the polynomial $Q(x) = (x + a_1)(x + a_2)^3 + A_1x + A_2x^2$ has four real roots. Let $Q_1(x) = (x + a_1)(x + a_2)^3$ and $Q_2(x) = A_1x + A_2x^2$. Let $-c \neq a_2$ be the root of the polynomial $Q'_1(x)$ and let $-d$ be the root of $Q'_2(x)$.

If $d < c$, then Q is decreasing on $(-\infty, -c]$, so Q has at most one root in this interval. Therefore Q has at least 3 roots in the interval $(-c, 0)$. Thus $Q''(x)$ has a root in the interval $(-c, 0)$, which is impossible, since $Q''(x) > 0$ on $(-c, 0)$.

If $a_2 \geq d \geq c$, then Q is increasing on the interval $[-c, 0)$ and decreasing on the interval $(-\infty, -d]$, so Q must have at least two roots in the interval $(-d, -c)$. But $Q(x) < 0$ on this interval.

Finally, if $d > a_2$, then Q may only have roots in the union $(-\infty, a_2) \cup (-a_1, 0)$. But Q is increasing on $(-a_1, 0)$, so Q has 3 roots in $(-\infty, a_2)$. This however is impossible, since $Q''(x) > 0$ for $x \in (-\infty, a_2)$. Thus $\{2, 3\} \subseteq S$ and the polynomial

$$(x + a_1)(x + a_2)^3 + tx^2(x + a_2)$$

has, for all sufficiently small $t > 0$, four distinct roots.

4) Since the polynomial $(x + a_1)^2(x + a_3)^2 + A_2x^2$ has no real roots, $1 \in S$ or $3 \in S$. Then the polynomial $(x + a_1)^2(x + a_3)^2 + tx^k$ for $k = 1, 3$ has, for all sufficiently small $t > 0$, four distinct real roots.

5) In view of the above remarks, $S = \{1, 2, 3\}$. Consider

$$r(x) = (x + a_1)^4 + tx^3 + 2ta_1x^2 + t(a_1^2 - t^2)x = (x + a_1)^4 + tx((x + a_1)^2 - t^2).$$

Then for all sufficiently small $t > 0$, $a_1^2 - t^2 > 0$, and the polynomial r has four distinct real roots, because

$$r(-a_1 - 2t) = t^3(10t - 3a_1) < 0, \quad r(-a_1) = a_1t^3 > 0 \quad \text{and} \quad r(-a_1 + 2t) = t^3(22t - 3a_1) < 0.$$

Thus we have proved:

Corollary 3.7

Conjecture 3.3 is true if $2 \leq n \leq 4$ and S is an arbitrary nonempty subset of $\{1, 2, \dots, n-1\}$.

This implies that the sum of squared logarithms inequality (Conjecture 1.2) holds also for $n = 4$.

Corollary 3.8 (Sum of squared logarithms inequality for $n = 4$)

Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 > 0$ be given positive numbers such that

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &\leq b_1 + b_2 + b_3 + b_4, \\ a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4 &\leq b_1 b_2 + b_1 b_3 + b_2 b_3 + b_1 b_4 + b_2 b_4 + b_3 b_4, \\ a_1 a_2 a_3 + a_1 a_2 a_4 + a_2 a_3 a_4 + a_1 a_3 a_4 &\leq b_1 b_2 b_3 + b_1 b_2 b_4 + b_2 b_3 b_4 + b_1 b_3 b_4, \\ a_1 a_2 a_3 a_4 &= b_1 b_2 b_3 b_4. \end{aligned}$$

Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 + \log^2 a_4 \leq \log^2 b_1 + \log^2 b_2 + \log^2 b_3 + \log^2 b_4.$$

Proof. Use Corollary 3.7 and observe that S may be an arbitrary subset of $\{1, 2, 3\}$. ■

Corollary 3.9

Let $n \geq 2$ be an integer and let T be an arbitrary subset of $\{1, 2, \dots, n-1\}$. Assume that the Conjecture 3.3 holds for n and for any nonempty subset S of T . Let moreover $f \in C^n(0, \infty)$. Then the inequality

$$f(a_1) + f(a_2) + \dots + f(a_n) \leq f(b_1) + f(b_2) + \dots + f(b_n)$$

holds for all $a, b \in \Delta_n$ satisfying

$$E_k(a) \leq E_k(b) \quad \text{for } k \in T \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k = 0 \text{ or } k \notin T \quad (3.8)$$

if and only if

$$(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0 \quad \text{for all } x > 0 \text{ and all } k \in T. \quad (3.9)$$

Proof. Assume first (3.9) holds and let $a, b \in \Delta_n$ satisfy (3.8). Consider any $c \in \Delta_n$ with $a \leq c < b$. Then the pair c, b satisfies condition (2.4) for some nonempty subset S of T . Therefore by Proposition 3.4, the set $\mathcal{C}(c, b)$ is nonempty and hence by Proposition 3.2, $a \preceq b$. Now Theorem 2.3 implies that inequality (2.6) holds.

Conversely, if (2.6) holds for all $a, b \in \Delta_n$ satisfying (3.8), then (2.6) also holds for all $a, b \in \Delta_n$ satisfying condition (2.4) with $S = T$. Thus Theorem 2.4 implies (3.9). This completes the proof. \blacksquare

4 Outlook

Our result generalizes and extends the previously known results on the sum of squared logarithms inequality. Indeed, compared to the proof in [1] our development here views the problem from a different angle in that it is not the logarithm function that defines the problem, but a certain monotonicity property in the geometry of polynomials, explicitly stated in Conjecture 3.3.

If one tries to adopt the above proof of Conjecture 3.3 for $n \leq 4$ to the case $n \geq 5$, one has to deal with approximately 2^n cases considered separately. Therefore it is clear, that the extension to natural numbers n beyond $n = 6$, say, is out of reach with such a method. Instead, a general argument should be found to prove or disprove Conjecture 3.3 for general n . Furthermore, it might be worthwhile to develop a better understanding of the differential inequality condition $(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed fully to all parts of this paper.

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References

- [1] M. Bîrsan, P. Neff, and J. Lankeit. Sum of squared logarithms - An inequality relating positive definite matrices and their matrix logarithm. *J. Inequalities Appl.* 2013, DOI: 10.1186/1029-242X-2013-168:2013:168, 2013.
- [2] F.M. Dannan, P. Neff, and C. Thiel. On the sum of squared logarithms inequality and related inequalities. *submitted*, 2015. arXiv:1411.1290.

- [3] J. Lankeit, P. Neff, and Y. Nakatsukasa. The minimization of matrix logarithms - on a fundamental property of the unitary polar factor. *J. Lin. Alg. Appl.*, 449:28–42, 2014.
- [4] P. Neff, B. Eidel, F. Osterbrink, and R. Martin. A Riemannian approach to strain measures in nonlinear elasticity. *C. R. Acad. Sci. Paris (Mecanique)*, 342(4):254–257, 2014.
- [5] P. Neff, I.D. Ghiba, J. Lankeit, R. Martin, and D. Steigmann. The exponentiated Hencky-logarithmic strain energy. Part II: Coercivity, planar polyconvexity and existence of minimizers. *to appear in Z. Angew. Math. Phys.*, 2014. arXiv:1408.4430v1.
- [6] P. Neff, J. Lankeit, and I.D. Ghiba. The exponentiated Hencky-logarithmic strain energy. Part I: Constitutive issues and rank-one convexity. *to appear in J. Elast.*, 2014. arXiv:1403.3843.
- [7] P. Neff, J. Lankeit, and A. Madeo. On Grioli’s minimum property and its relation to Cauchy’s polar decomposition. *Int. J. Engng. Sci.*, 80:209–217, 2014.
- [8] P. Neff, Y. Nakatsukasa, and A. Fischle. A logarithmic minimization property of the unitary polar factor in the spectral norm and the Frobenius matrix norm. *SIAM J. Matrix Analysis*, 35:1132–1154, 2014. arXiv:1302.3235v4.